quent equations of system (7) by, respectively, the expressions

$$\frac{d\theta_2}{d\theta_1} = \frac{\lambda_2}{\lambda_1}\left(1 + \frac{1}{\lambda_2 r_2}\Theta_2\right)\left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1}$$

$$\frac{d\theta_n}{d\theta_1} = \frac{\lambda_n}{\lambda_1}\left(1 + \frac{1}{\lambda_n r_n}\Theta_n\right)\left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1}$$

system (7) is transformed to

$$\frac{du_i}{d\theta_1} = \frac{1}{\lambda_1} U_i \left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1}$$

$$\frac{dr_1}{d\theta_1} = \frac{1}{\lambda_1} R_1 \left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1}$$

$$\frac{dr_n}{d\theta_1} = \frac{1}{\lambda_1} R_n \left(1 + \frac{1}{\lambda_1 r_1}\Theta_1\right)^{-1} \tag{8}$$

In system (8) we then average (over a single period) those coefficients that are functions of $\cos\theta_s$ and $\sin\theta_s$ and which have a period 2π. After this the system assumes the final form

$$\frac{du_i}{d\theta_1} = \frac{1}{\lambda_1} \bar{U}_i + [U_i] \qquad \frac{dr_s}{d\theta_1} = \frac{1}{\lambda_1} \bar{R}_s + [R_s] \tag{9}$$

where the terms of higher orders are included in $[U_i]$ and $[R_s]$.

System (9) describes the case when $(m + n)$ zero roots have $2(m + n)$ groups of solutions.

Thus, the investigation of the stability of the motion described by system (1) with periodic coefficients (2) is reduced to an investigation of the stability of the motion described by system (9) with constant coefficients, when the latter system has $2(m + n)$ groups of solutions for $(m + n)$ zero roots. This case was examined in Ref. 3, and, on the basis of investigations of system (9), it is possible to conclude that G. V. Kamenkov's theorems on the stability or instability of the motion described by system (1) remain in force, if the periodic coefficients (2) of system (1) are bounded functions of t, and if the parameter ϵ does not exceed the value following from Sylvester's conditions.

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Mathematical Simulation of the Dynamics of Certain Processes of Fluidization

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A method is proposed for setting up a pattern for mathematical simulation of the dynamics of certain processes in apparatus with fluidized beds, based on the description of the macro-kinetics of heterogeneous processes in such beds, by use of multidimensional phase space. In this type of space, the heterogeneous process is considered a "pseudohomogeneous" one. Some examples illustrating the application of the proposed method are presented.

ONE of the interesting regions in which methods of mathematical simulation are applied is the investigation of the dynamics of different processes in equipment with fluidized beds. For simulation of the dynamics of such processes, one must first of all construct a sufficiently general mathematical model to include within its scope both homogeneous and heterogeneous processes. In many cases it is the heterogeneous processes which determine the transient behavior of equipment.

The present paper is an attempt to solve the problem indicated by assuming that heterogeneous processes in fluidized beds may be considered "pseudohomogeneous" processes in multidimensional phase space. So far as the authors know, the literature on equipment in which fluidized beds exist is concerned only with the dynamics of truly homogeneous processes.[1]

Phase Space and Zones of Homogeneity

In order possibly to develop an all-inclusive scheme for description of homogeneous and heterogeneous processes, it is convenient to use the idea of phase space in describing the distribution of solid particles. This space is defined in terms of a set of rectangular axes along which we measure different quantities characterizing a particle—for example, the three-space coordinates, the degree of transformation, or any other characteristic of the material, temperature, etc. Each state of the process would correspond in phase space to a particle representing the probability function, and these particles in the aggregate would correspond to the density of a fluid filling all parts of phase space.

Using this description of processes, we shall encounter both real liquids and gases in three-dimensional space and also certain fictitious liquids in multidimensional phase spaces. In this description, the dynamics of a process reduce simply to the motion of the given liquids.

It should be noticed, however, that the states of the liquids in the spaces considered do not determine their motion

uniquely, since these states do not contain information on the distribution of the momenta. (From this viewpoint, it would be more accurate to use the term "configuration space." However, we prefer the designation "phase," because at each point of the given space the particles undergo, in general, a specified phase transformation.)

To avoid this difficulty, we assume that the essential characteristics of the change in a flow process with fluidized beds may be characterized adequately by certain numbers (for example, Re, Ar, etc.), formed from various averaged quantities characterizing the processes. In this situation we do not require, generally speaking, the introduction of individual numbers for all apparatus; instead, we presume only that such numbers may be introduced, for particular zones in a given setup. There are many data in the literature indicating that such an assumption is proper. Using this approach, a closed description of the dynamics of a process may be obtained by introducing empirical correlation criteria in a general system of equations.

For the purpose of mathematical simulation, we introduce a similar idea of zones of homogeneity of a process—that is, we refer to such parts of phase space (including ordinary space), throughout which the probability density of particles (that is, the density of the fictitious fluid) and the density and temperature of real liquids or gases may be considered constant with an adequate degree of precision. In this manner, the problem of simulation reduces to simulating the dynamics of a process in each zone of homogeneity and to discovering the relation between these zones.

In the following we illustrate the application of the proposed method.

Phase Space of Zero Dimension

We consider certain thermodynamic processes in which several different materials—solid and gaseous—are involved. The description of the process by use of phase space of zero dimension corresponds to the following idealization:

1) The process is homogeneous throughout the apparatus; that is, the particles and gas are thoroughly mixed.

2) The mean composition of the particles involved in the process remains constant in the different working regimes of the apparatus.

The kinetic equations of the process may be described in the following general form:[2]

$$v_i = v_i(c_1, \ldots, c_n; T, \Gamma)$$
$$v_T = v_T(c_1, \ldots, c_n; T, \Gamma) \tag{1}$$

where v_i and v_T are, respectively, the amount of the ith material and of heat dissipated (or generated) per unit volume

per unit time as a consequence of chemical reactions; c_i the mass concentration of the ith material; T the temperature; Γ a set of quantities (numbers) determining the state of motion in the apparatus; and n the number of materials.

Assuming that the processes are homogeneous, we do not recognize in Eqs. (1) any difference between homogeneous and heterogeneous reactions, and in all cases we consider the mass concentration of the material.

The requirements of material and thermal balance have the form

$$dG_i/dt = Q_i - Q_i' - v_i V \tag{2}$$

$$dG_T/dt = Q_T - Q_T' - v_T V \tag{3}$$

where G_i and G_T are the amount of the ith material and the amount of heat in the apparatus; Q_i and Q_T the flow of the ith material and the flow of heat entering the apparatus; Q_i' and Q_T' the flows leaving the apparatus; V the volume of the fluidized bed; and t the time:

$$G_i = V c_i \tag{4}$$

$$G_T = TV \sum_{(i)} c_i \mu_i \tag{5}$$

where μ_i is the specific heat of the ith material.

In this approximation, we express the quantities determining the nature of the motion in the apparatus as functions of the outward flows, the temperature, and the flows of material dissipated or generated in the apparatus. The dependence of these quantities on the composition of particles is not considered since it is assumed constant. For simplicity we shall assume that the weight of the layer is constant, which, generally speaking, is not necessary. Thus, we write in the general case

$$\Gamma = \Gamma(Q_1, \ldots, Q_n; V v_1, \ldots, V v_n; T) \tag{6}$$

The currents of material leaving the bed depend on the concentration of material and on the quantities determining the flow regime. This may be written

$$Q_i' = Q_i'(c_i, \Gamma) \tag{7}$$

The heat flow out of the bed depends on the currents of material and on the temperature of the layer:

$$Q_T' = Q_T'(Q_1', \ldots, Q_n'; T) \tag{8}$$

(We neglect the dynamical effect of heat accumulation by the walls of the apparatus and assume a constant external temperature).

In Fig. 1 the mathematical simulation of the dynamics of the process taking account of Eqs. (1–8) includes $(n + 1)$
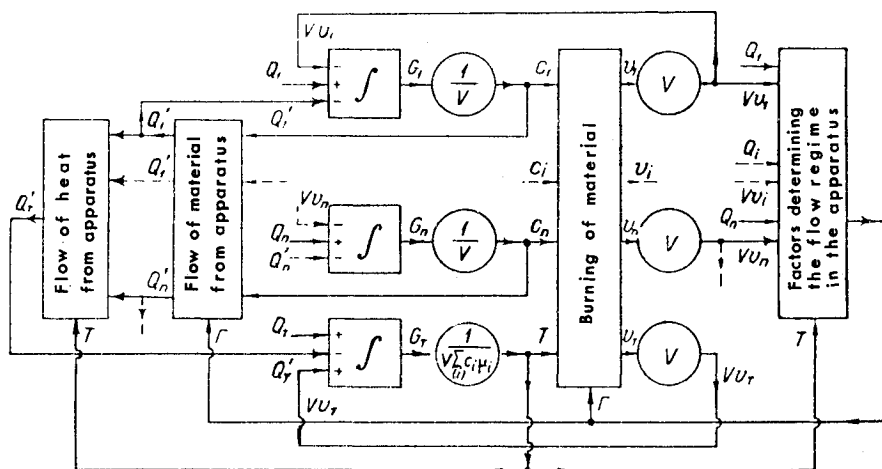

Fig. 1   Simulation of dynamics of processes in a fluidized bed, taking account of Eqs. (1–8)

integrating blocks according to the number of equations of material and heat conservation and inertia-free functional blocks. In each specific case the simulation could be substantially simplified by isolating those features of the dynamics which exhibit the most interest (see, for example, Fig. 2).

In Fig. 2 it is assumed that the reaction depends on a process of the first order, and that the currents of material out of the bed are proportional to the concentration of these materials in the bed. K_A and K_B are coefficients of proportionality. The diagram of this figure corresponds to an equation of the chemical reaction having the form

$$aA + bB = cC + dD \tag{9}$$
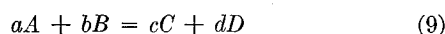
where A, B, C, and D designate the materials. Masses of material are measured in gram-molecules or in gram-atoms; a, b, c, and d are stoichiometric coefficients.

If, in Fig. 2, we consider the thermal balance and introduce the relations expressing the effect of temperature change on the rate constant of the reaction, we can simulate the dynamics of the process with varying temperature. A more compli-
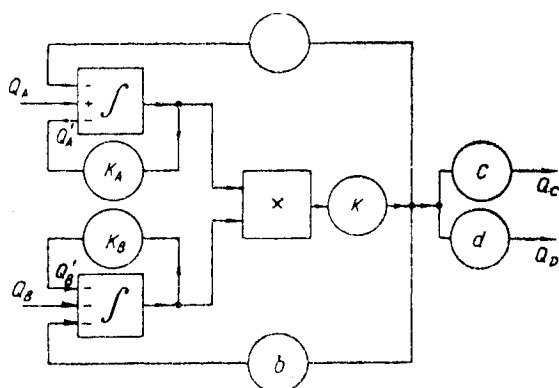

Fig. 2. Simulation of a process including only one chemical reaction, for regimes stabilized in temperature and hydrodynamical relations

cated diagram would include the influence of hydrodynamic factors. The scheme described here, of course, remains applicable only if the assumption is valid that the system may be expressed in a zero-dimensional phase space—that is, if we consider the process to be homogeneous as previously described.

One-Dimensional Phase Space

To illustrate an application of the ideas of a one-dimensional phase space, we consider the burning process of the particles of any material in a fluidized bed. We shall assume as before that the process is completely one-dimensional in the apparatus space, an assumption that would not be valid in any three-dimensional problem. However, in contrast with the foregoing case, we do not now assume constancy in the average fractional composition of the particles.

To make the problem more definite, we introduce the following assumptions, which have no fundamental significance but which make the problem more easily visualized.

1) The burning reaction occurs on the surface of the particles and is of the first order with respect to oxygen. (The equation of the reaction may be assumed to have the form of (9), where, for example, B designates oxygen.)

2) In the burning process, the particles change only in radius, retaining their spherical shape.

3) The current of material flowing out of the bed is proportional to the quantity of material in the bed; the coefficient of proportionality is assumed to depend on the radius of the particles.

For simplicity we consider only regimes which have stabilized temperature and hydrodynamical relationships, since the influence of temperature and hydrodynamic factors can be accounted for quite adequately in complete analogy with the preceding section by using a supplementary diagram and by introducing the necessary connections. Thus we bring into the computation only the requirement of material conservation, and assume that the constant of macroscopic reaction rate has a magnitude dependent only on the radius of the particles. Under these conditions, the phase space introduced has only a single coordinate, namely, the radius r of the particles.

The rate of change of the radius of the individual particles may be written

$$dr/dt = -KC_B \tag{10}$$

where K is the constant of the burning rate and C_B is the concentration of oxygen.

The quantity dr/dt is simply the rate of change of the coordinate r, that is, the velocity of the fictional fluid previously referred to along a given direction in phase space. Thus we obtain a complete analogy with the phenomenon of convective transport, so that the component of the flow along the r axis has the form

$$q^r = \rho \, (dr/dt) \ = -K\rho C_B \tag{11}$$

where ρ is the number of particles existing per unit radius.

Taking into account assumption 3 of this section and expression (11), the equation of conservation for the number of particles may be written

$$\partial\rho/\partial t = KC_B \, \mathrm{div}\rho + q' - K_A\rho \tag{12}$$

where q' is the density of sources of particles and K_A is the proportionality coefficient between the number of particles in the layer and the number of particles removed from the layer in unit time.

The operation div in this case designates simply $\partial/\partial r$.

The rate of change of the mass of a particle is related to the rate of change of its radius by

$$\frac{dm}{dt} = \frac{d}{dt}\left(\frac{4}{3}\,\pi r^3\gamma\right) = 4\pi r^2\gamma\,\frac{dr}{dt} \ = -4\pi\gamma KC_B r^2 \tag{13}$$

where γ is the density of the particle material.

The quantity of material B expended in burning, per unit of time—Q_B'—is determined by

$$Q_B' = -\frac{b}{a}\,\int_0^{r_{max}} \frac{dm}{dt}\,\rho dr = \frac{b4\pi\gamma k}{a}\,C_B\int_0^{r_{max}} r^2\rho dr \tag{14}$$

where b/a is the ratio of the stoichiometric coefficients from formula (9). The equation of material conservation for the material B has the form

$$\frac{dG_B}{dt} = Q_B - Q_B' - Q_\Gamma'\,\frac{RT}{pV}\,G_B$$

$$Q_\Gamma' = Q_\Gamma - Q_B' \tag{15}$$

where Q_Γ and Q_Γ' are, respectively, the flow of gas into the bed and out of it; p the mean pressure in the bed; V the volume of gas in the bed; T the temperature; and G_B the mass of the material B in the bed.

Thus the problem of reproducing the dynamics of the process will reduce simply to the problem of simulating Eqs. (12), (14), and (15). This system of equations may be solved approximately, for example, by use of an analogue computer, if one goes from a continuous coordinate r to a system of lumped circuits, that is, by the assumption of zones of homogeneity. The approximate equations (12) and (14) are then written

$$\begin{cases} \dfrac{d\rho_1}{dt} = \dfrac{K}{\Delta r_1} C_B(\rho_2 - \rho_1) + q_1' - K_A\rho_1 \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ \dfrac{d\rho_n}{dt} = -\dfrac{K}{\Delta r_n} C_B\rho_n + q_n' - K_A\rho_1 \end{cases} \quad (16)$$

$$Q_B' \simeq \sum_{i=1}^{n} K_i C_B \rho_i \qquad \bar{K}_i = \frac{4\pi\gamma K b r_i{}^2 \Delta r_i}{a} \qquad (17)$$

The diagram for simulation of Eqs. (15–17) is shown in Fig. 3. It is interesting to note that in addition to the effects described by Eqs. (15–17) the same circuitry can be used to simulate the process of degradation of the particles; for this purpose, it is necessary to assign a coefficient of degradation K_{iz} and to introduce the relation between the zones of homogeneity with different radii. One of the relations is shown by the dotted line.

At the expense of complicating the scheme, we might wish to account for the influence of temperature and hydrodynamic factors; if so, then within the scope of the approximation already made, we might reproduce the selective degradation of particles and the different effects of temperature and hydrodynamic factors by use of constants $K_1, \ldots, K_n$, corresponding to particles of different radii.

Phase Space of Several Dimensions

Phase space of several dimensions might be introduced by considering several one-dimensional effects at the same time. For example, if it were necessary in the process described in the preceding section simultaneously to account for distribution of particles as to size and of any coordinate of dimension of the apparatus, we would change to a two-dimensional phase space. If the space coordinate were, for example, the depth of the fluidized bed x, then, as shown in Refs. 1–3, we might express the components of the flow along that coordinate as

$$q^x = -D\,\mathrm{grad}\rho \qquad \left(\mathrm{grad} = \frac{d}{dx}\right) \qquad (18)$$

where D is the mixing coefficient.

The component of flow along the r axis will be determined as before by Eq. (11).

The condition of material conservation is written

$$d\rho/dt = -\mathrm{div}q + q' \qquad (19)$$

where q' is the density of sources of particles, and $\mathrm{div} = d/dx + d/dr$.

If the space coordinate is the depth of the apparatus, then it is especially convenient to use the assumption of zones of uniformity. Thus, dividing the apparatus along the depth direction into a series of sections within each of which uniformity may be assumed, and using some convenient assumption of the mechanism of exchange between these portions, it is possible to carry out the simulation of the dynamic properties of the process. By considering the second coordinate r, we may include effects as, for example, the burning up of the fine particles above the bed.

With the development of the simulation technique, the solutions of similar and considerably more complicated problems will become more accessible to many research workers. Therefore, the method proposed in this paper for describing the dynamics of processes in apparatus with fluidized beds may prove to be very convenient and practicable for setting up suitable simulation methods. A detailed review of the existing simulating equipment may be found, for example, in Ref. 4.

Use of Mathematical Models for Analysis of Kinetic Experiments

If a sufficiently complete description of the dynamics of any particular process is available, mathematical models may be applied successfully as a convenient procedure for processing experimental data.

For a series of kinetic experiments, reactors with fluidized beds[5] are widely used at present. Results of experiments in such reactors reveal a variety of effects, and as a consequence the question arises of what information should be extracted from the experimental material. Ordinarily, experimental data is processed by a mathematical technique which is applied with confidence only to very simple effects. Since these descriptions often only partially exhibit the physical properties of the phenomena, their use often lacks the generality desired and cannot serve as a reliable and quantitative indication of the nature of the process under investigation. Therefore, a more detailed mathematical description of the phenomenon is required. The treatment of experimental data by use of ordinary methods can be done only with extreme difficulty, so that simulation equipment appears to be especially convenient.

In this paper it has been shown that mathematical models permit the synthesis of effects characterizing complex experi-

Fig. 3   Approximate simulation of Eqs. (15–17)

ments from relatively simple effects. Since, in this, the model carries out all necessary computations automatically, it follows that, by its use, the reverse problem should also be solvable—that is, the problem of analyzing experimental material. If the values of coefficients are selected on the model to guarantee the best agreement with experiment, we can extract in the process the values of a whole series of constants from the experimental material. Through developing a series of experiments in this fashion, it may be possible to obtain relations reflecting the influence of different factors on the given constants. Such an approach, according to our conclusions, may be considered especially useful, since experimental research has the objective of gaining information necessa y for designing industrial equipment.

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Reviewer's Comment

This paper contributes to the literature a procedure of mathematical simulation of dynamics in the fluidized solids equipment by means of an analog computer. Several schemes of mathematical simulation are given for some simple cases.

Provided that the mass transfer and chemical reaction follow the simple models assumed in this paper, the proposed mathematical simulation is acceptable. However, there are a number of other factors that can control the overall transfer or reaction rates in the fluidized beds. For instance, discussion of the formation of bubbles, dispersion of fine solid particles in the bubbles, contact efficiency between the gas stream and solid particles, and back mixing, etc., is lacking in this paper. Therefore, readers should take note of the restricted assumptions, employed here for the establishment of the fundamental equations.

Only the diagrams of simulation of the proposed fundamental equations are presented in this paper, and the reviewer thought it would be more valuable if it had included several results of numerical calculation, preferably as the dimensionless charts. Then the reader could compare his experience using the fluidized solids equipment with the mathematical treatments proposed in this paper or could evaluate the adequacy of the fundamental assumptions.

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Determination of the Probability of Loss of Stability by a Shell

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THE calculation of stability of a shell is the establishment of a critical value λ_* of the loading parameter λ such that for $\lambda < \lambda_*$ the shell is guaranteed to be in a state considered acceptable for operation. According to the theory of stability of shells (e.g., Ref. 1), a recommended value for λ_* is the upper critical loading λ_+, calculated for shells that are ideal in the sense that imperfections of form, material, and the like may be ignored in its computation. As a basis for this we may reason that the unperturbed state is stable in the sense of Liapunov only for $\lambda < \lambda_+$. However, comparison of theoretical data with the results of experiments[2] shows that the actual λ_* lies significantly below the value computed from classical theory. At present this fact is explained[3-5] by the observation that the stability of shells is significantly influenced by the distribution of characteristics of the shell on the one hand and by perturbations of the loading on the other.

In the present note we wish to indicate that the influence of the forementioned groups of factors has a different character.

Let the characteristics of the shell and the nature of its support be described by a system of parameters $a_j (j = 1, 2, \ldots, m)$ which are random variables with joint probability density function $\varphi(a_j)$, known from experiment. Let

the one-dimensional probability density function of the loading λ be denoted by $\psi(\lambda)$. In the general case $\psi(\lambda)$ depends on the time; that is, λ is a random process. Let us assume, following Refs. 3 and 5, that the spectral density of $\lambda(t)$ is different from zero only in the frequency range $\omega \ll \omega_0$, where ω_0 is the lowest natural frequency of oscillation of the shell. At $t = t_0$ we put the shell in an unperturbed equilibrium state with zero initial velocity and then observe its evolution under the influence of a random loading $\lambda(t)$. If other factors of a dynamical character are absent, then it follows from our assumptions that this evolution will be through a sequence of equilibrium states, and, from the definition of λ_+, that in each separate case loss of stability occurs at $\lambda = \lambda_+(a_j)$. One may conclude that the distribution of geometric and physical properties of the shell, the method of support, and the loadings for its quasi-static motion reduce simply to the corresponding distribution of the upper critical loading. The possibility of loss of stability for $\lambda_-(a_j) < \lambda < \lambda_+(a_j)$, where λ_- is the lower critical loading, depends on dynamical perturbing factors having energy sufficient to overcome the potential barrier. Thus the method of Bolotin[3,5] (in that part where the concept "quasi-static" is used) may be considered a means of studying the distribution of the upper critical loading.

From the preceding reasoning there follows a method of determining the probability of loss of stability, somewhat different from that described in Refs. 3 and 5. Under the assumption of a quasi-static loading, we may write

$$P(*|a_j, \lambda) = \sigma_0[\lambda - \lambda_+(a_j)] \qquad (1)$$